3 次元非圧縮性
渦なし流れ

非圧縮性完全流体流れの解析例

完全流体、渦無し、バロトロピー流体、保存力

次に示す解では非圧縮、保存外力ゼロ、定常流

●ラプラス方程式（連続方程式）

$$\Delta\Phi = 0$$

●渦無し条件

$$\mathbf{u} = grad\Phi$$

●ベルヌーイの式

$$\frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} = const$$

ラプラス方程式の解

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

1. 一様流

$$\frac{\partial^2 \Phi}{\partial x^2} = 0$$

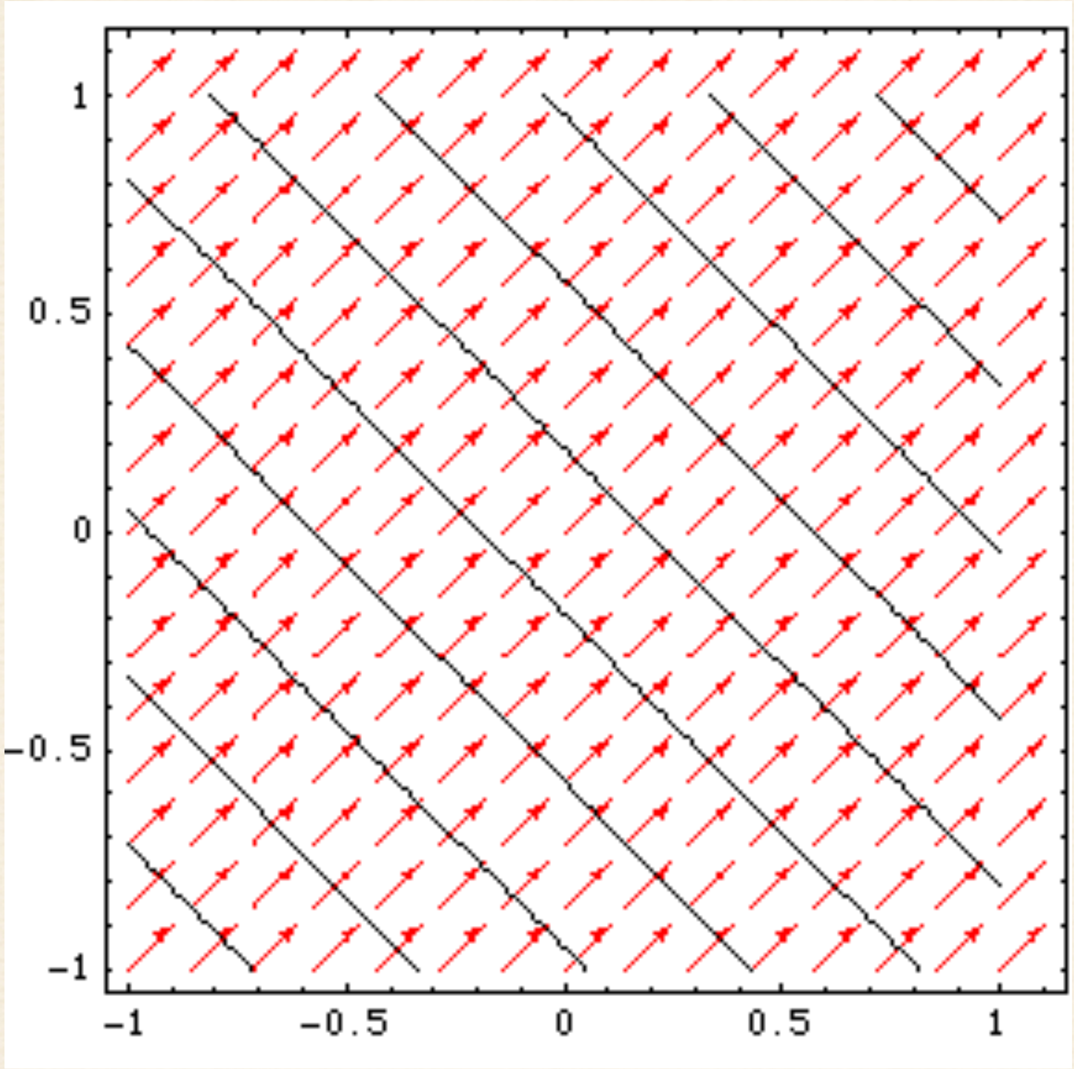
$$\frac{\partial \Phi}{\partial x} = u$$

$$\Phi = ux$$

$$p = const \quad \Leftarrow \quad \frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} = const$$

一般解　$\Phi = ux + vy + wz$　　$p = const$

$$\Phi = x + y$$



実線は速度ポテンシャルの等値線

ラプラス方程式の解

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Phi}{\partial \theta}\right)$$

$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial \phi^2}=0$$

3次元極座標表現

2.わき出し、吸い込み

球対称性から θ 方向 ϕ 方向は一様であるため、θ および ϕ 微分はゼロ

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r} \right) = 0$$

$m>0$

わき出し

$$r^2 \frac{\partial \Phi}{\partial r} = m \quad \longrightarrow \quad v_r = \frac{m}{r^2}$$

$m<0$

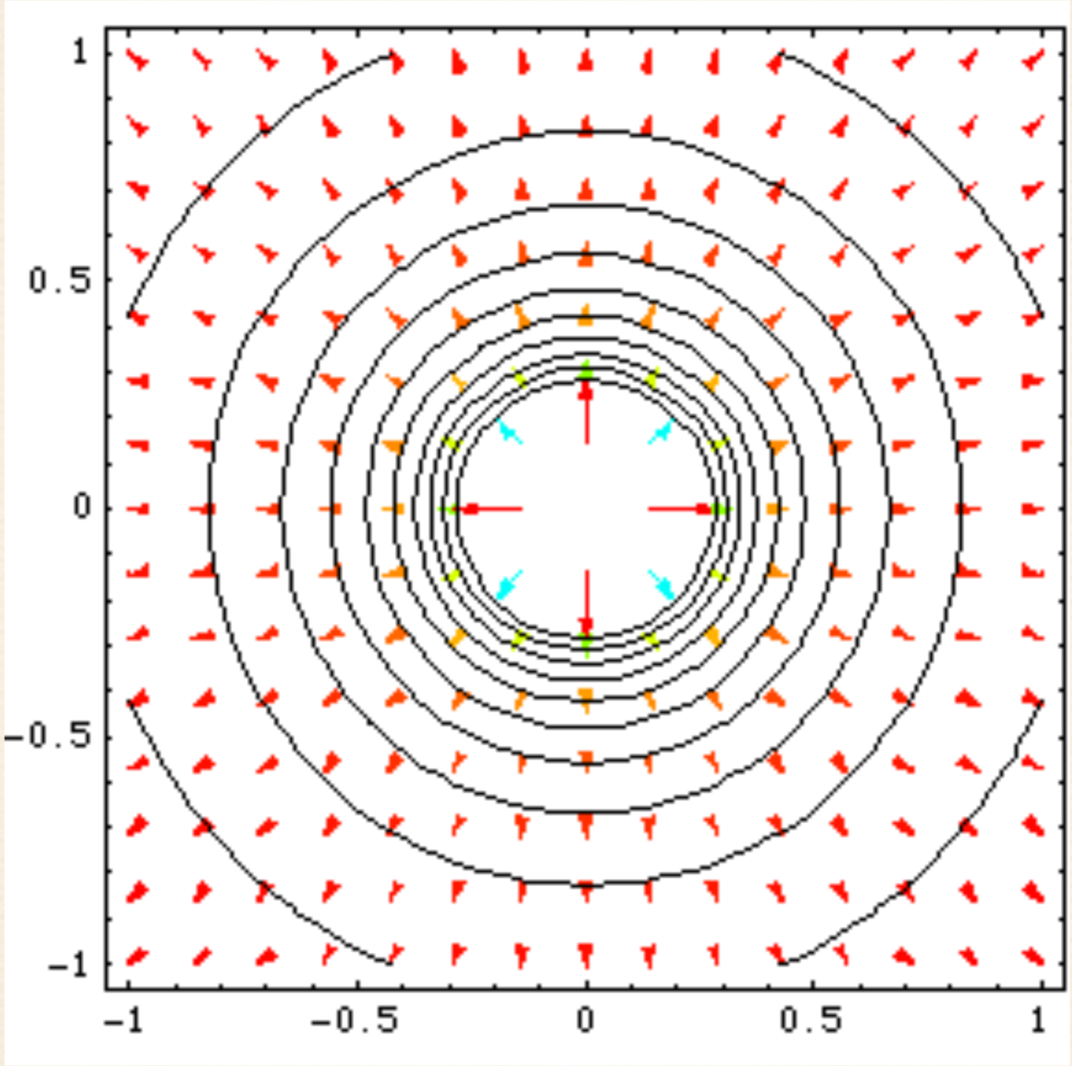
吸い込み

$$\Phi = -\frac{m}{r} + c$$

$$p = p_\infty - \frac{1}{2}\rho \frac{m^2}{r^4} \quad \longleftarrow \quad \frac{1}{2}\left|\mathbf{u}\right|^2 + \frac{p}{\rho} = const$$

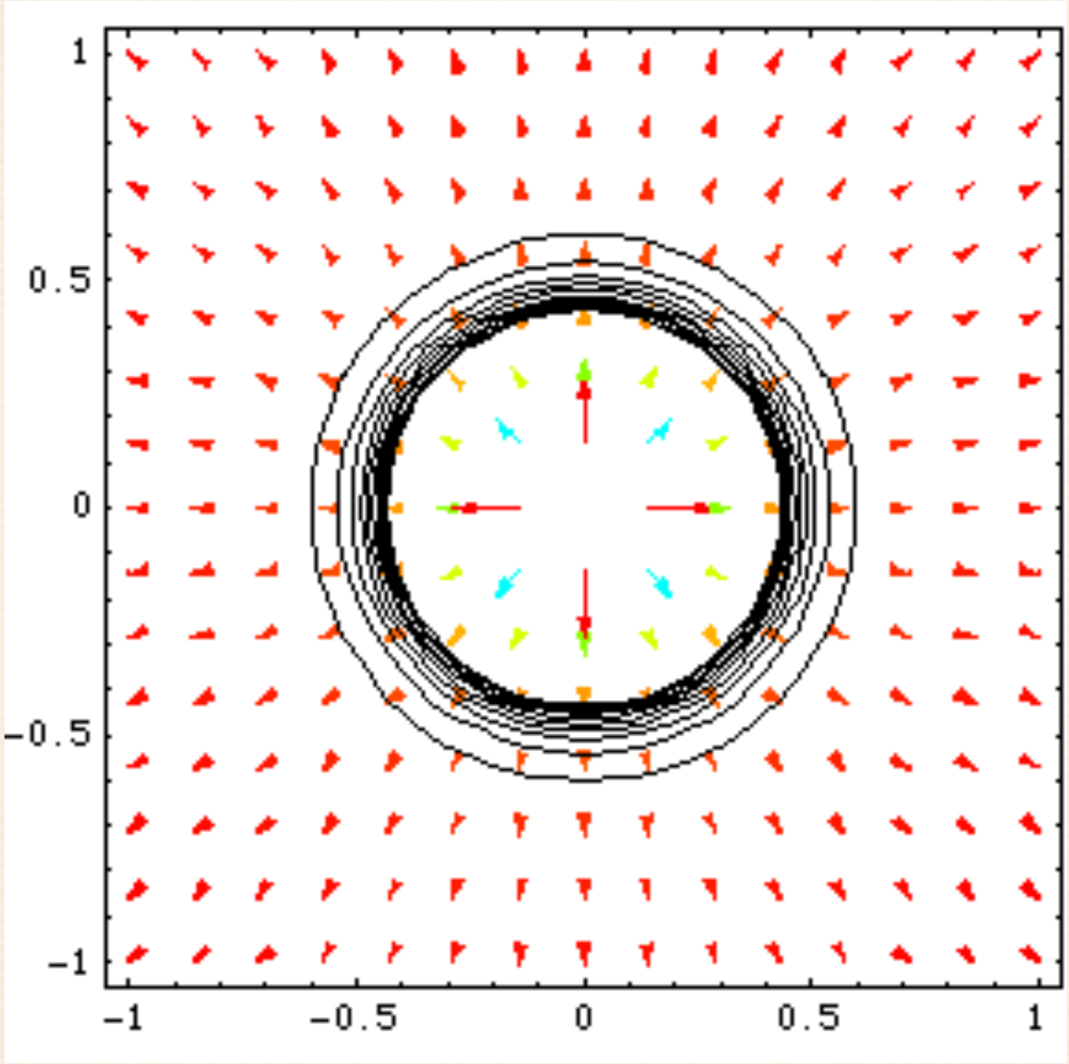
$$\frac{1}{2}v_r^2 + \frac{p}{\rho} = \frac{1}{2}0^2 + \frac{p_\infty}{\rho}$$

$$\Phi = -\frac{1}{\sqrt{x^2 + y^2 + z^2}}, z = 0$$



実線は速度ポテンシャルの等値線

$$p = -\frac{1}{\left(x^2 + y^2 + z^2\right)^2}, z = 0$$



実線は圧力の等値線

ラプラス方程式の解

既に解いた2つの解をラプラス方程式の線形性を利用して、解を作成

3.半無限体

一様流とわき出しの解の線形結合

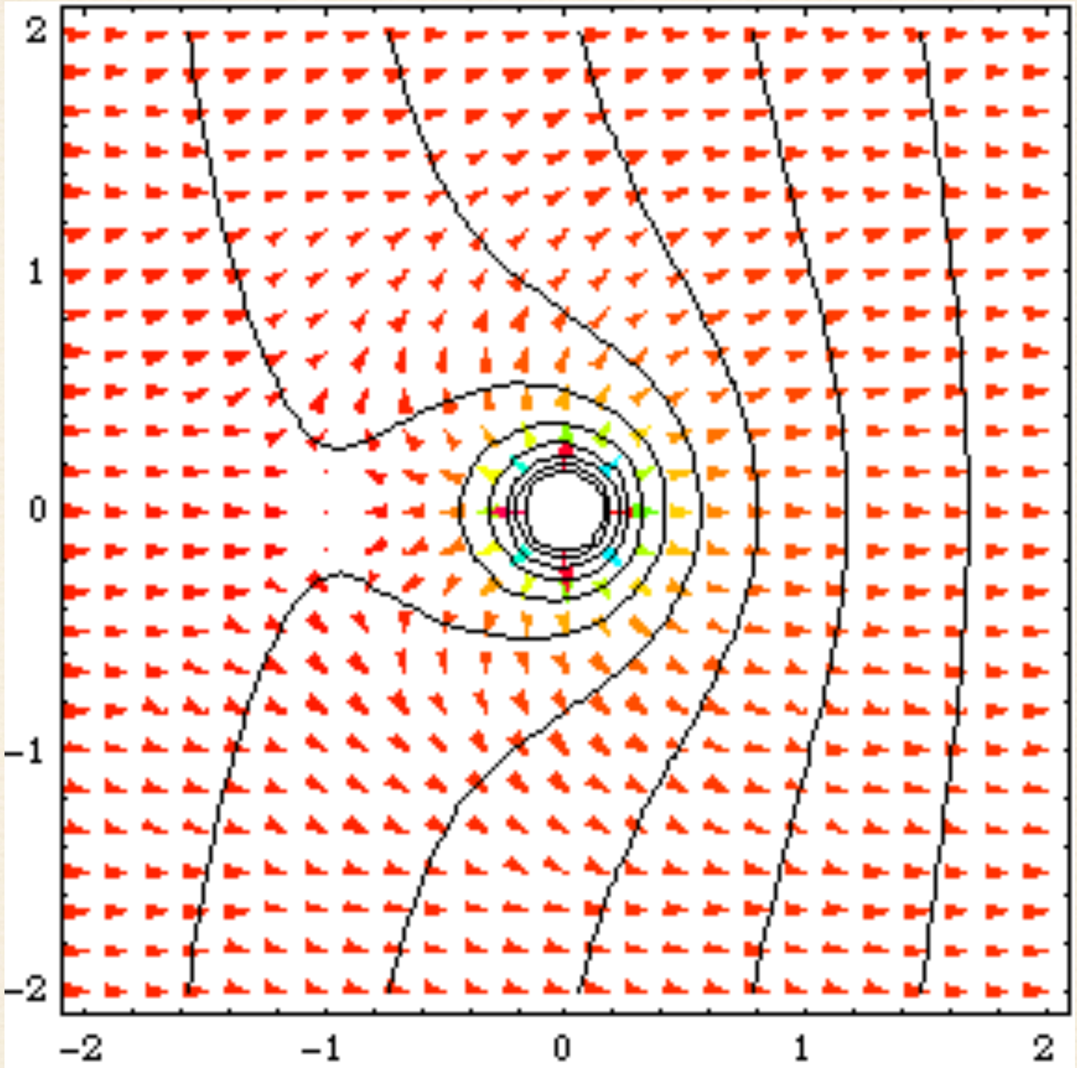
$$\Phi = Ux - \frac{m}{r}$$

$$u = \frac{\partial \Phi}{\partial x} = U + \frac{mx}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

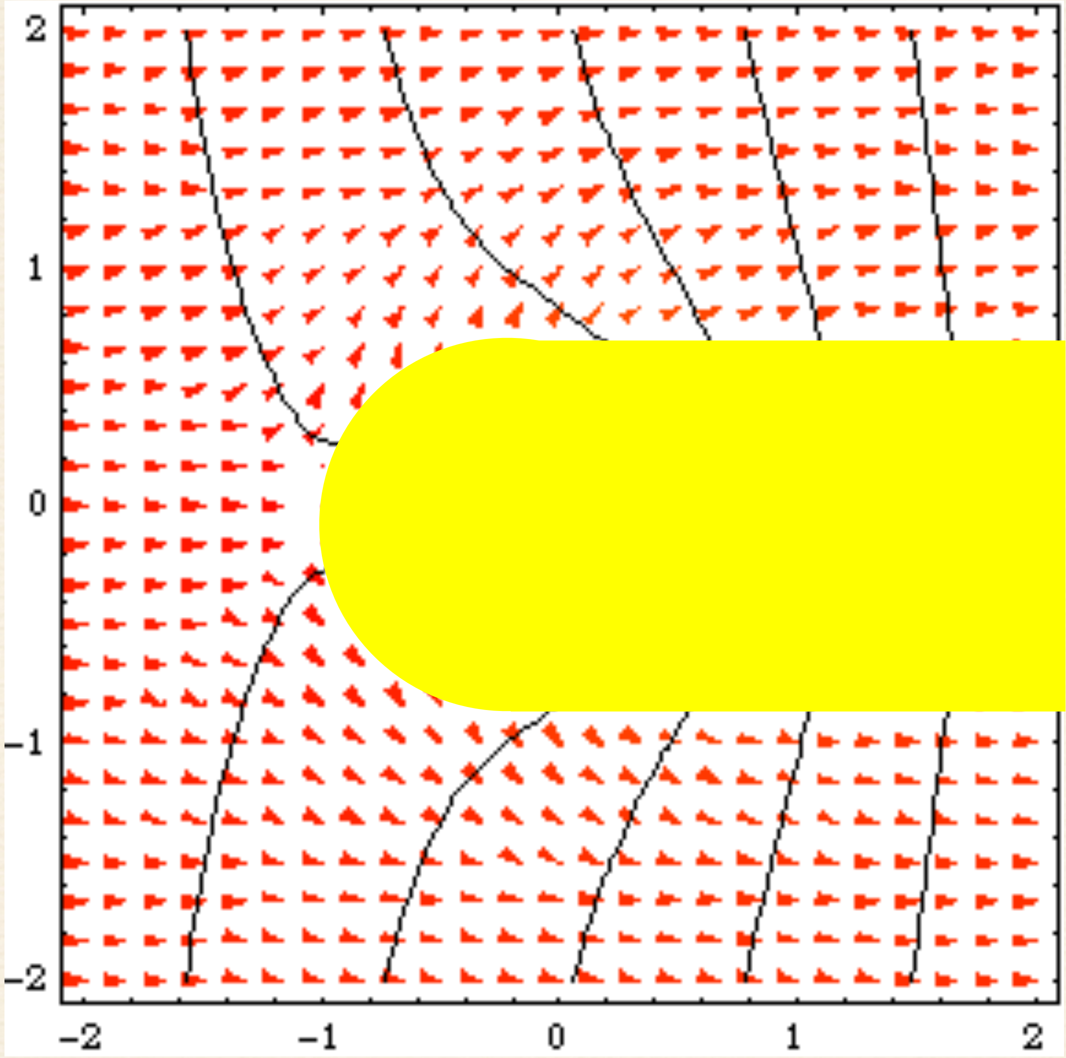
$$v = \frac{\partial \Phi}{\partial y} = \frac{my}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$w = \frac{\partial \Phi}{\partial z} = \frac{mz}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$\Phi = x - \frac{1}{\sqrt{x^2 + y^2 + z^2}}, z = 0$$

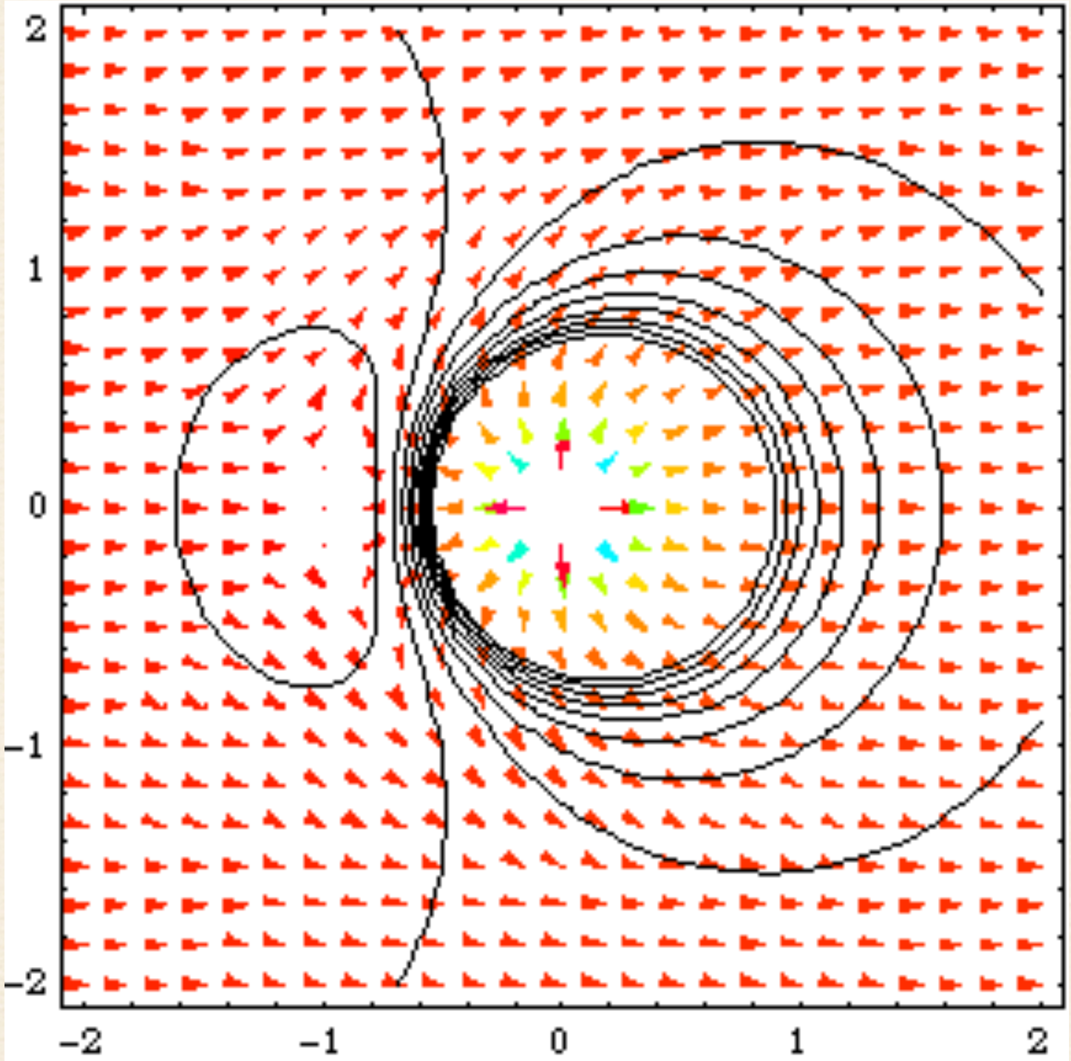


実線は速度ポテンシャルの等値線



実線は速度ポテンシャルの等値線

$$\frac{p}{\rho}=\frac{p_\infty}{\rho}-\frac{1}{2}\frac{m^2}{\left(x^2+y^2+z^2\right)^2}-\frac{1}{2}\frac{2Umx}{\left(x^2+y^2+z^2\right)^{3/2}}$$



実線は圧力の等値線

ラプラス方程式の解

4. ランキン卵形

一様流とわき出しと吸い込みの解の線形結合

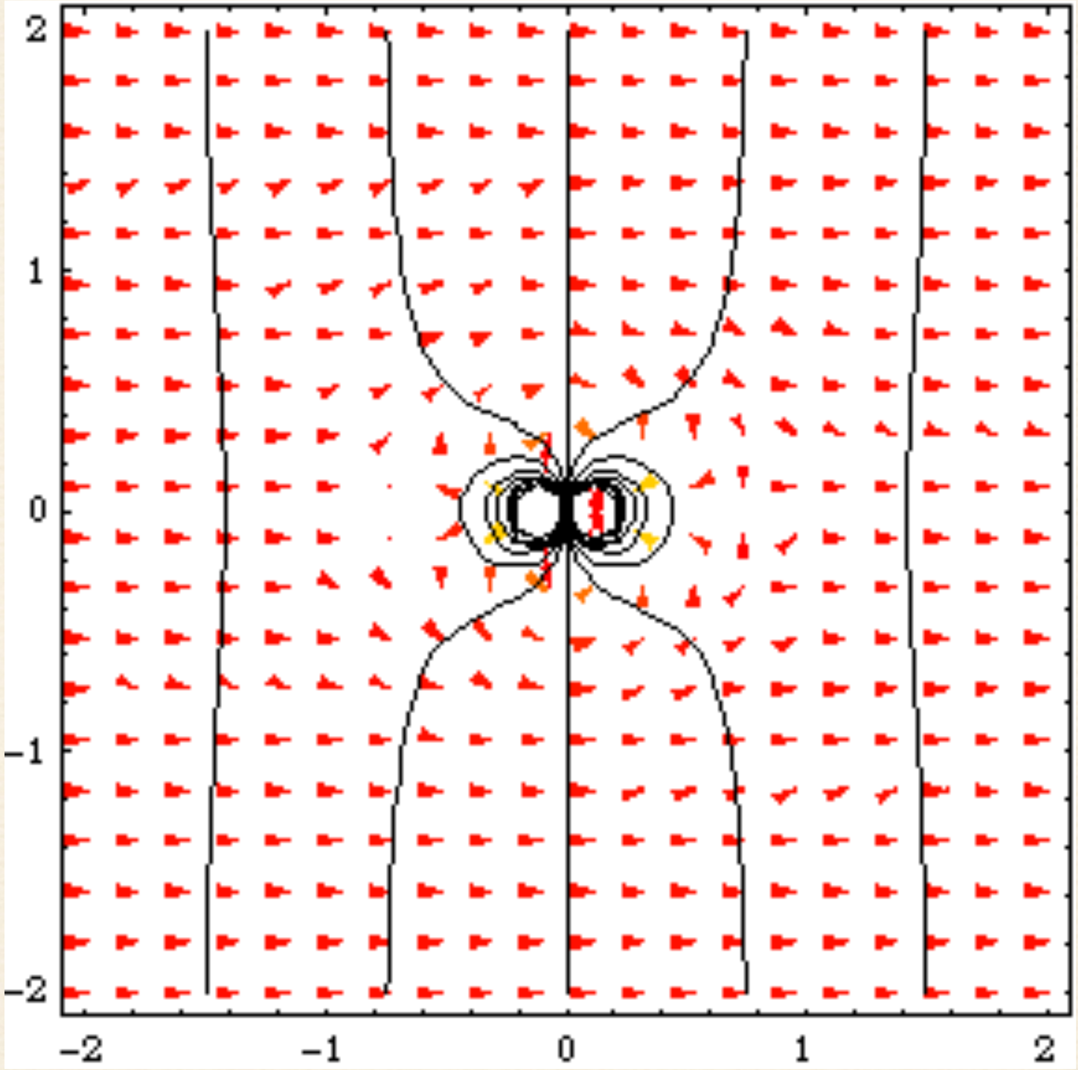
$$\Phi = Ux - \frac{m}{r_1} + \frac{m}{r_2}$$

$$u = \frac{\partial \Phi}{\partial x} = U + \frac{m\left(x + x_0\right)}{\left(\left(x + x_0\right)^2 + y^2 + z^2\right)^{3/2}} - \frac{m\left(x - x_0\right)}{\left(\left(x - x_0\right)^2 + y^2 + z^2\right)^{3/2}}$$

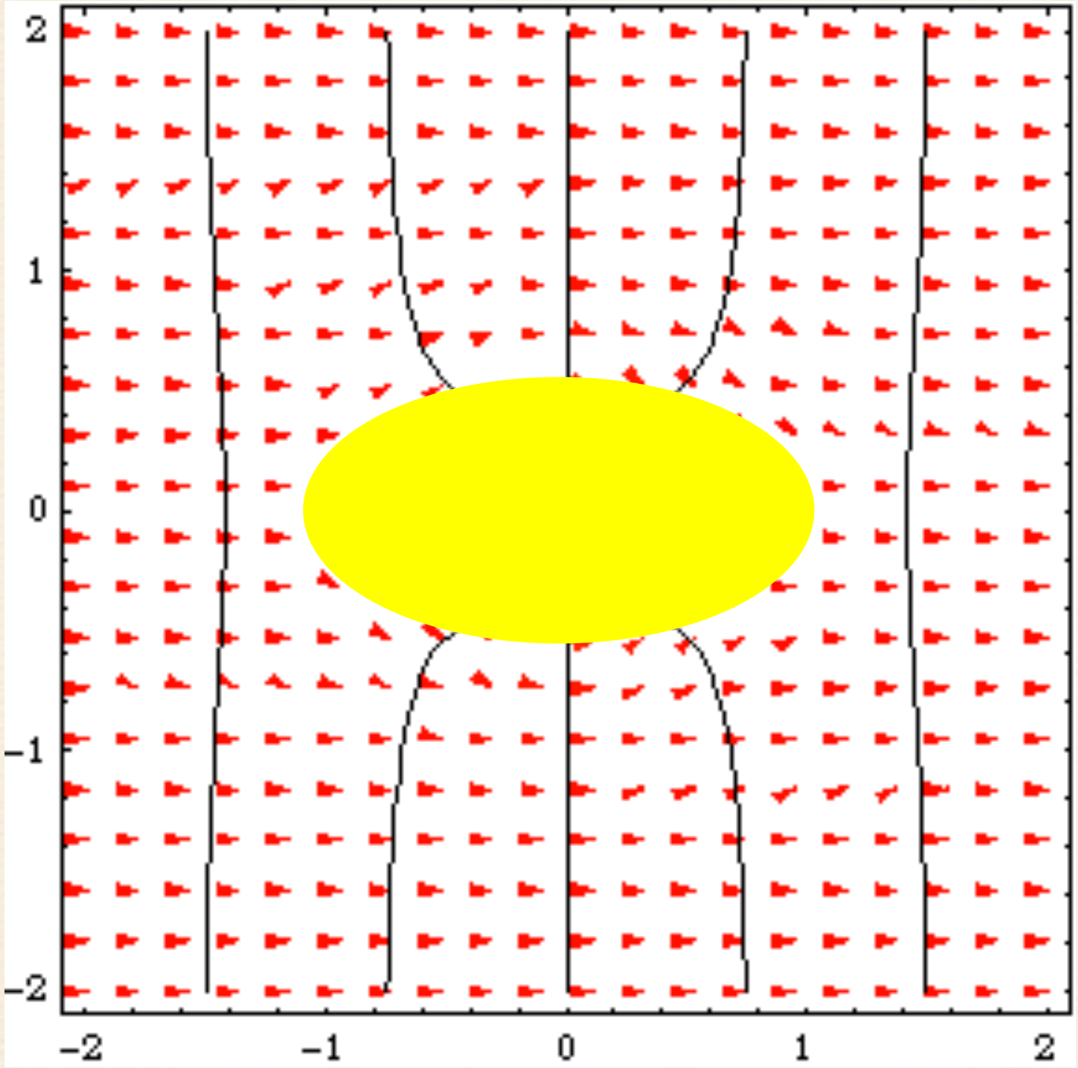
$$v = \frac{\partial \Phi}{\partial y} = \frac{my}{\left(\left(x + x_0\right)^2 + y^2 + z^2\right)^{3/2}} - \frac{my}{\left(\left(x + x_0\right)^2 + y^2 + z^2\right)^{3/2}}$$

$$w = \frac{\partial \Phi}{\partial z} = \frac{mz}{\left(\left(x + x_0\right)^2 + y^2 + z^2\right)^{3/2}} - \frac{mz}{\left(\left(x + x_0\right)^2 + y^2 + z^2\right)^{3/2}}$$

$$\Phi = x - \frac{1}{\sqrt{(x+0.1)^2 + y^2 + z^2}} + \frac{1}{\sqrt{(x-0.1)^2 + y^2 + z^2}}, z = 0$$



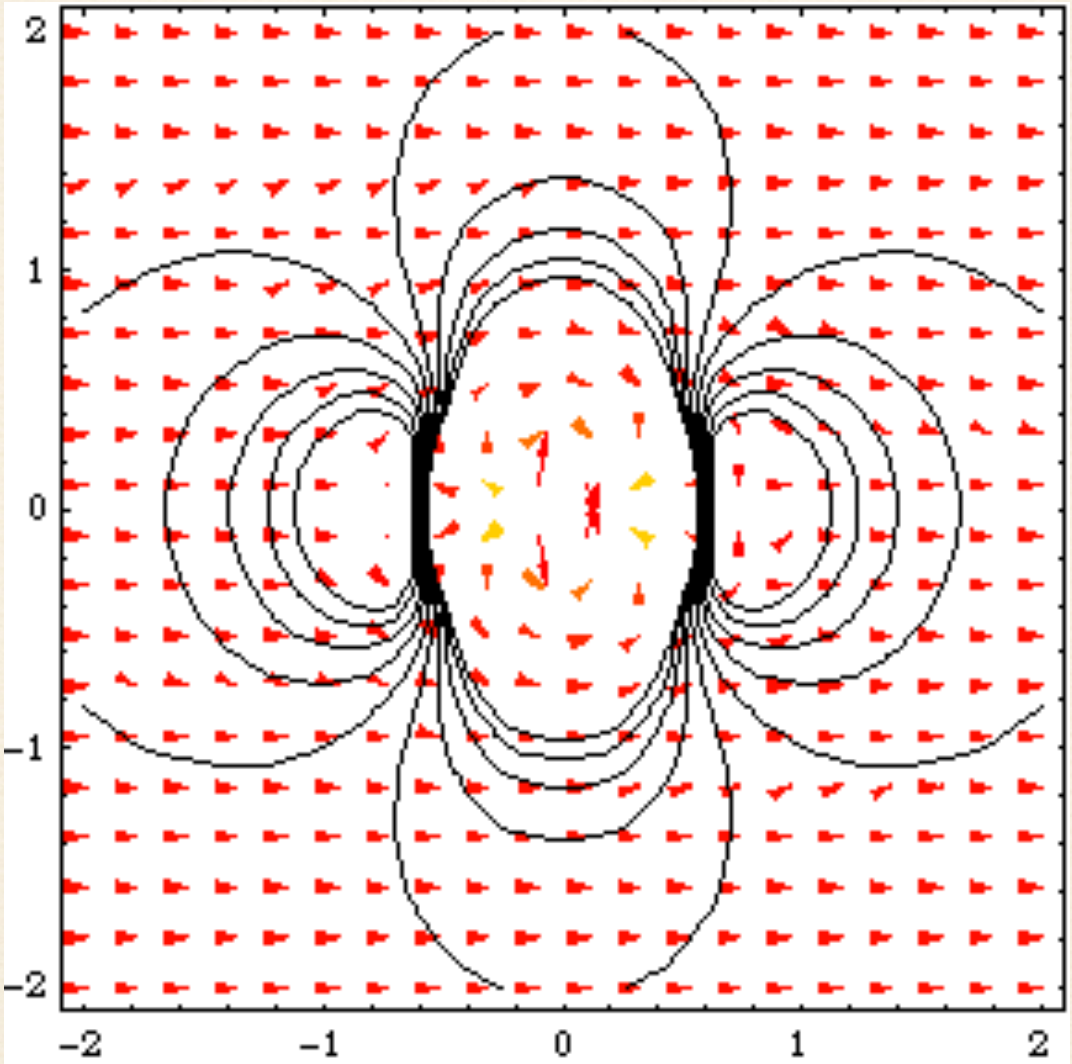
実線は速度ポテンシャル



実線は速度ポテンシャルの等値線

$$\frac{p}{\rho}=\frac{p_\infty}{\rho}-\frac{1}{2}\frac{m^2}{\left(\left(x+x_0\right)^2+y^2+z^2\right)^2}-\frac{1}{2}\frac{m^2}{\left(\left(x-x_0\right)^2+y^2+z^2\right)^2}-\frac{Um\left(x+x_0\right)}{\left(\left(x+x_0\right)^2+y^2+z^2\right)^{3/2}}$$

$$+\frac{Um\left(x-x_0\right)}{\left(\left(x-x_0\right)^2+y^2+z^2\right)^{3/2}}+\frac{m^2\left(x+x_0\right)\left(x-x_0\right)+m^2y^2+m^2z^2}{\left(\left(x+x_0\right)^2+y^2+z^2\right)^{3/2}\left(\left(x-x_0\right)^2+y^2+z^2\right)^{3/2}}$$



実線は圧力の等値線

ラプラス方程式の解

5.二重わき出し吸い込み

双極子

わき出しと吸い込みの解の線形結合でその
ソース点を無限に近づける

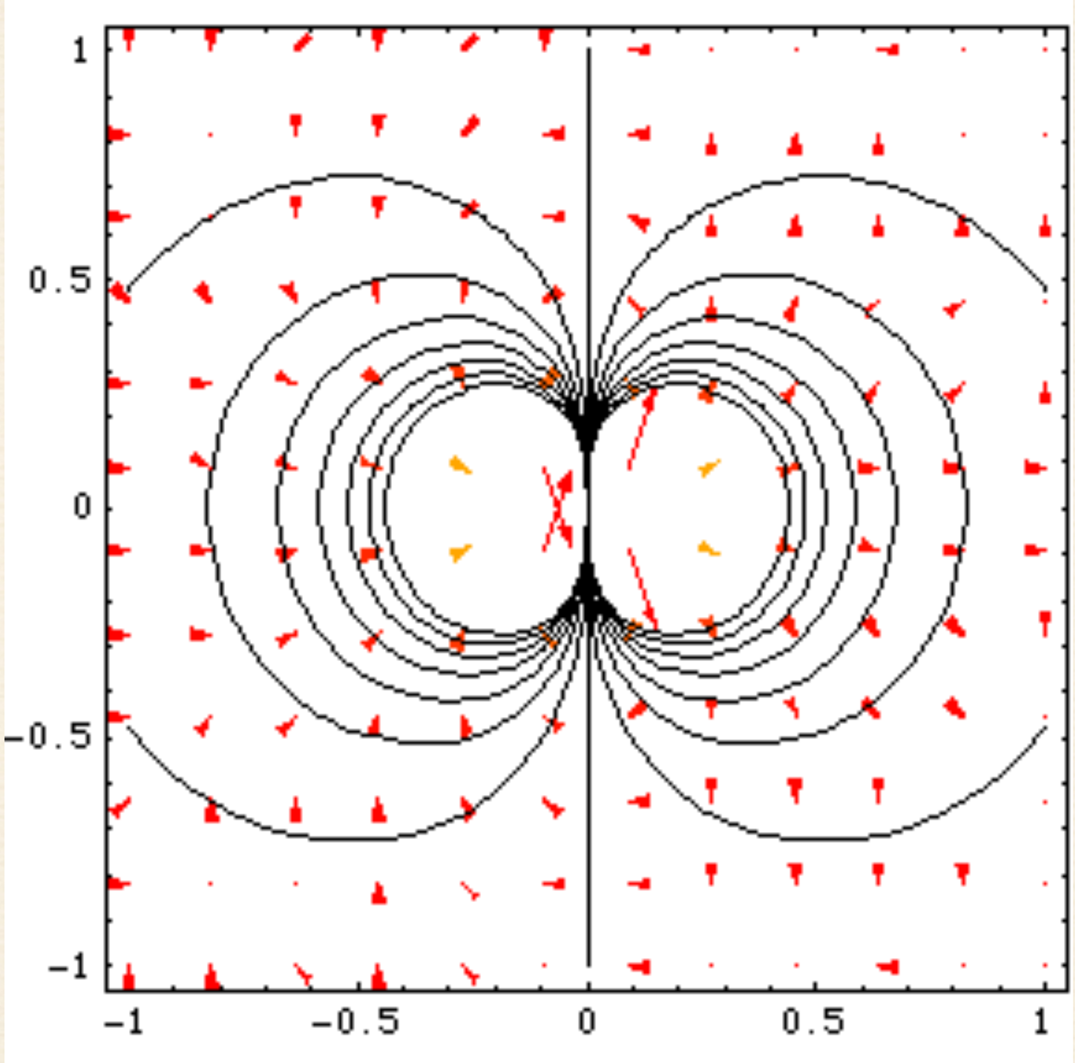
$$\Phi = \lim_{OO_1 \to 0} \left(\frac{m}{r_1} - \frac{m}{r_2} \right) = -\mu \frac{\cos\theta}{r^2}$$

$$u = \frac{\partial \Phi}{\partial x} = -\frac{\mu}{\left(x^2 + y^2 + z^2\right)^{3/2}} + \frac{3\mu x^2}{\left(x^2 + y^2 + z^2\right)^{5/2}}$$

$$v = \frac{\partial \Phi}{\partial y} = \frac{3\mu xy}{\left(x^2 + y^2 + z^2\right)^{5/2}}$$

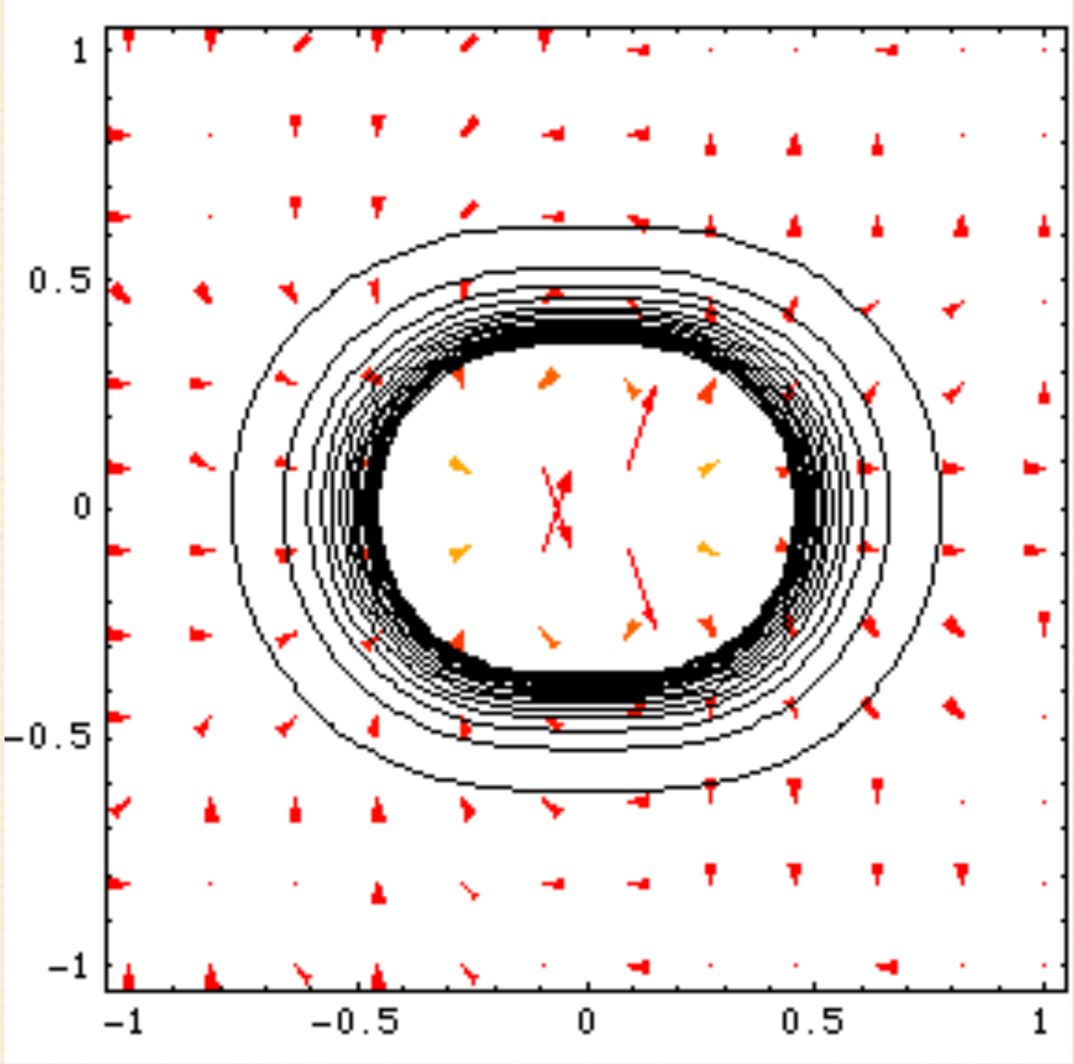
$$w = \frac{\partial \Phi}{\partial z} = \frac{3\mu xz}{\left(x^2 + y^2 + z^2\right)^{5/2}}$$

$$\Phi = -\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}},\ z=0$$



実線は速度ポテンシャル

$$\frac{p}{\rho} = \frac{p_\infty}{\rho} - \frac{1}{2}\frac{\mu^2}{\left(x^2+y^2+z^2\right)^3} - \frac{1}{2}\frac{3\mu^2 x^2}{\left(x^2+y^2+z^2\right)^4}$$



実線は圧力の等値線

ラプラス方程式の一般解

デカルト直交系

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

変数分離 $\Phi(x,y,z) = \Xi(x)\Psi(y)Z(z)$

$$\frac{1}{\Xi(x)}\frac{\partial^2 \Xi(x)}{\partial x^2} + \frac{1}{\Psi(y)}\frac{\partial^2 \Psi(y)}{\partial y^2} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} = 0$$

解析解

$$\Phi(x,y,z) = \left(A_x^+ e^{ik_x x} + A_x^- e^{-ik_x x}\right)\left(A_y^+ e^{ik_y y} + A_y^- e^{-ik_y y}\right)\left(A_z^+ e^{ik_z z} + A_z^- e^{-ik_z z}\right)$$

$$-k_x^2 - k_y^2 - k_z^2 = 0$$

円筒座標系

$$\frac{\partial^2 \Phi}{\partial r^2}+\frac{1}{r}\frac{\partial \Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}+\frac{\partial^2 \Phi}{\partial z^2}=0$$

変数分離 $\Phi(r,\theta,z)=R(r)\Theta(\theta)\mathrm{Z}(z)$

$$\frac{1}{R(r)}\frac{\partial^2 R(r)}{\partial r^2}+\frac{1}{r}\frac{1}{R(r)}\frac{\partial R(r)}{\partial r}+\frac{1}{r^2}\frac{1}{\Theta(\theta)}\frac{\partial^2 \Theta(\theta)}{\partial \theta^2}+\frac{1}{\mathrm{Z}(z)}\frac{\partial^2 \mathrm{Z}(z)}{\partial z^2}=0$$

解析解

$$\Phi(r,\theta,z)=\left(A_1 J_n(kr)+B_1 N_n(kr)\right)\left(A_2\sin n\theta+B_2\cos n\theta\right)\left(A_3 e^{kz}+B_3 e^{-kz}\right)$$

$$J_n(\xi)=\sum_{m=0}^{\infty}\frac{(-1)^m \xi^{n+2m}}{2^{n+2m} m!\Gamma(m+n+1)},\quad N_n(\xi)=\frac{1}{\sin n\pi}\left(\cos n\pi J_n(\xi)-J_{-n}(\xi)\right)$$

Bessel関数 Neumann関数

3次元極座標系

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

変数分離 $\Phi(r,\theta,\varphi)=R(r)\Theta(\theta)\vartheta(\varphi)$

$$\frac{\sin^2\theta}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right)+\frac{\sin\theta}{\Theta(\theta)}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right)+\frac{1}{\vartheta(\varphi)}\frac{\partial^2\vartheta(\varphi)}{\partial\varphi^2}=0$$

解析解

$$\Phi(r,\theta,\varphi)=\left(A_n r^n+B_n r^{-n-1}\right)\left(C_n^m P_n^m(\cos\theta)+D_n^m Q_n^m(\cos\theta)\right)\left(E_m e^{im\varphi}+F_m e^{-im\varphi}\right)$$

$P_n^m(\cos\theta)$, $Q_n^m(\cos\theta)$ 陪ルジャンドル関数